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Study of Lotka-volterra food chain chemostat with periodically varying dilution rate

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In this paper, we introduce and study a model of Lotka-volterra chemostat food chain chemostat with periodically varying dilution rate, which contains with predator, prey, and substrate. We investigate the subsystem with substrate and prey and study the stability of the periodic solutions, which are the boundary periodic solutions of the system. The stability analysis of the boundary periodic solution yields an invasion threshold. By use of standard techniques of bifurcation theory, we prove that above this threshold there are periodic oscillations in substrate, prey, and predator. Simple cycles may give way to chaos in a cascade of period-doubling bifurcations. Furthermore, we numerically simulate a model with sinusoidal dilution rate, by comparing bifurcation diagrams with different bifurcation parameters, we can see that the system experiences following process: periodic solution → periodic doubling cascade →chaos.

KEY WORDS: bifurcation, Lotka-volterra functional response, chemostat, periodic dilution, chaos

1. Introduction and the model

As well known, countless organisms live in seasonally or diurnally forced environment, in which the populations obtain food, so the effects of this forcing may be quite profound. There is evidence, for example, the seasonal variation in

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contact rates derives the dynamics of childhood disease epidemics [1], and that seasonal or diurnal periodicity in competition coefficients can play a pivotal role in the coexistence of some competitors [2]. A chemostat is a common laboratory apparatus used to culture microorganisms. Sterile growth medium enters the chemostat at a constant rate; the volume within the chemostat is held constant. In its simplest form, the system approximates conditions for plankton growth in lakes, where the limiting nutrients such as silica and phosphate are supplied from streams draining the watershed. Recently many papers studied chemostat model with variations in the supply of nutrients or the washout. Chemostat with periodic inputs are studied in [3–7], those with periodic washout rate in [8, 9], and those with periodic input and washout in [10]. In this paper, we introduce and study a model of Lotka-volterra type food chain chemostat with periodically varying dilution rate, we may write

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}T} &= D(1+\varepsilon A(T))(S_0 - S) - \tfrac{\mu_1}{\delta_1} SH,\\
\frac{\mathrm{d}H}{\mathrm{d}T} &= \mu_1 SH - D(1+\varepsilon A(T))H - \tfrac{\mu_2}{\delta_2} HP,\\
\frac{\mathrm{d}P}{\mathrm{d}T} &= \mu_2 HP - D(1+\varepsilon A(T))P,
\end{aligned} \tag{1.1}$$

where is the τ/D-period continuous function, with

$$\int_0^{\frac{\tau}{D}} A(t)\mathrm{d}t = 0, \quad |A(t)| \leqslant 1.$$

The state variables S, H, and P represent the concentration of limiting substrate, prey, and predator. D is the dilution rate; μ_1 and μ_2 are the uptake and predation constants of the prey and predator; δ_1 is the yield of prey per unit mass of substrate; δ_2 is the biomass yield of predator per unit mass of prey; b_1, b_2 are half capturing saturation constants of prey and predator. DS_0 units of substrate are added, on average, per unit of time. $n \in N$, N is the set of all non-negative integers.

There are advantages in analyzing dimensionless equations. We treat the reciprocal of the dilution rate as natural measure of time:

$$x \equiv \frac{S}{S_0}, \quad y \equiv \frac{H}{\delta_1 S_0}, \quad z \equiv \frac{P}{\delta_1 \delta_2 S_0}, \quad t \equiv DT.$$

After some algebra, this yields

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= (1+\varepsilon a(t))(1-x) - m_1 xy,\\
\frac{\mathrm{d}y}{\mathrm{d}t} &= m_1 xy - (1+\varepsilon a(t))y - m_2 yz,\\
\frac{\mathrm{d}z}{\mathrm{d}t} &= m_2 yz - (1+\varepsilon a(t))z
\end{aligned} \tag{1.2}$$

with

$$m_1 = \frac{\mu_1 S_0}{D}, \quad m_2 = \frac{\mu_2 S_0}{D}, \quad a_1 = \delta_1 b_1, \quad a_2 = \delta_2 b_2.$$

The organizations of the paper are as following. In next section, we investigate the existence and stability of the periodic solutions of the impulsive subsystem with substrate and prey. In section 3, we study the locally stability of the boundary periodic solution of the system and obtain the threshold of the invasion of the predator. By use of standard techniques of bifurcation theory, we prove that above this threshold there are periodic oscillations in substrate, prey, and predator. In section 4, Furthermore, we numerically simulate a model with sinusoidal dilution rate, by comparing bifurcation diagrams with different bifurcation parameters, we can see that the system experiences following process: periodic solution $\rightarrow$ periodic doubling cascade $\rightarrow$chaos.

2. Behavior of the substrate bacterium subsystem

In the absence of the protozan predator, system (1.2) reduces to

$$\begin{aligned} \frac{dx}{dt} &= (1 + \varepsilon a(t))(1 - x) - m_1 xy, \\ \frac{dy}{dt} &= m_1 xy - (1 + \varepsilon a(t))y. \end{aligned} \tag{2.1}$$

This nonlinear system has simple periodic solutions. For our purpose, we present these solutions in this sections.

If we add the first and second equations of the system (2.1), we have $\frac{d(x+y)}{dt} = (1 + \varepsilon a(t))(1 - x - y)$. If we take variable changes $s = x + y$ then the system (2.1) can be rewritten as

$$\frac{ds}{dt} = (1 + \varepsilon a(t))(1 - s). \tag{2.2}$$

For the system (2.2), we have the following lemma 2.1.

Lemma 2.1. The subsystem (2.2) has a positive periodic solution $\tilde{s}(t)$ and for every solution $s(t)$ of (3.2) we have $|s(t) - 1| \rightarrow 0$ as $t \rightarrow \infty$.

Lemma 2.2. Let $(x(t), y(t))$ be any solution of system (2.1) with initial condition $x(0) \geqslant 0$, $y(0) > 0$, then $\lim_{t\to\infty} |x(t) + y(t) - 1| = 0$.

The lemma 2.2 says that the periodic solution $\tilde{s}(t)$ is uniquely invariant manifold of the system (2.1).

Theorem 2.1. For the system (2.1), we have that:

(1) If $m_1 < 1$, then the system (2.1) has a unique globally asymptotically stable boundary τ−periodic solution $(x_e(t), y_e(t))$, where

$$x_e(t) = 1, \quad y_e(t) = 0 \tag{2.3}$$

(2) If $m_1 > 1$, then the system (2.1) has a unique globally asymptotically stable positive τ−periodic solution $(x_s(t), y_s(t))$ and the τ−periodic solution $(x_e(t), y_e(t))$ is unstable. And we have

$$\frac{1}{\tau}\int_0^\tau y_s(l)\mathrm{d}l = \frac{m_1 - 1}{m_1}.$$

Proof.

(1) If $m_1 < 1$, it is obvious that

$$y(t) \leqslant y(0)\mathrm{e}^{(m_1-1)t}\exp\left(\int_0^t p_1(l)\mathrm{d}l\right), \tag{2.4}$$

where $p_1(t) = m_1\tilde{s}(t) - \int_0^\tau m_1\tilde{s}(l)\mathrm{d}l$; note that $\int_0^\tau p(l)\mathrm{d}l = 0$ and hence that $p_1(t)$ is τ-periodic continuous function. Thus, for $\frac{1}{\tau}\int_0^\tau m_1\tilde{s}(l)\mathrm{d}l - 1 < 0$ we find that $y(t)$ tends exponentially to zero as $t \to +\infty$. Consider the system (2.2), we have $x(t) = s(t) - y(t)$. By lemma 2.2, we have $\lim_{t\to\infty}|x(t) - \tilde{s}(t)| = 0$.

(2) Set $m_1 > 1$. By lemma 2.1, we can consider the system (2.1) in its stable invariant manifold $\tilde{s}(t)$, that is

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (m_1 - 1 - \varepsilon a(t))y - m_1 y^2. \tag{2.5}$$

The equation (2.5) has a globally asymptotically stable τ-period solution

$$y_s(t) = \left(\mathrm{e}^{(m_1-1)\tau} - 1\right)\left(\int_t^{t+\tau} m_1 \exp\left(-\int_s^t ((m_1 - 1 - \varepsilon a(l)))\mathrm{d}l\right)\mathrm{d}s\right)^{-1}$$

and $\frac{1}{\tau}\int_0^\tau y_s(t)\mathrm{d}t = \frac{m_1-1}{m_1}$.

We denote positive τ-periodic solution

$$x_s(t) = 1 - y_s(t).$$

For the system (2.1), by lemma 2.2 we obtain that for any solution $(x(t), y(t))$ with initial condition $x(0) \geqslant 0, y(0) > 0$, $|x - x_s| \to 0, |y - y_s| \to 0$ as $t \to \infty$. We complete the proof. □

3. The bifurcation of the system

In order to investigate the invasion of the predator of system (1.2), we add the first, second, and third equations of it and take variable changes $s = x+y+z$, by lemma 2.1, the following lemma is obvious.

Lemma 3.1. Let $(x(t), y(t), z(t))$ be any solution of system (1.2) with $X(0) > 0$, then

$$\lim_{t\to\infty} |x(t) + y(t) + z(t) - 1| = 0. \tag{3.1}$$

The lemma 3.1 says that the periodic solution $\tilde{s}(t)$ is an invariant manifold of the system (1.2).

Theorem 3.1. Let $(x(t), y(t), z(t))$ be any solution of system (1.2) with $X(0) > 0$.

(1) If $m_1 > 1$ and $m_2 < m_2^* := \frac{m_1}{m_1-1}$, then the system (1.2) has a unique globally asymptotically stable boundary τ-periodic solution $(x_s(t), y_s(t), 0)$ is globally asymptotical stable.

(2) If $m_1 > 1$ and $m_2 > m_2^* := \frac{m_1}{m_1-1}$, then the periodic boundary solution $(x_s(t), y_s(t), 0)$ of the system (1.2) is unstable.

Proof. The local stability of periodic solution $(x_s(t), y_s(t), 0)$ may be determined by considering the behavior of small amplitude perturbations of the solution. Define

$$x(t) = u(t) + x_s(t), \qquad y(t) = v(t) + y_s(t), \qquad z(t) = w(t)$$

there may be written

$$\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \Phi(t) \begin{pmatrix} u(0) \\ v(0) \\ w(0) \end{pmatrix}, \quad 0 \leqslant t \leqslant \tau,$$

where $\Phi(t)$ satisfies

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \begin{pmatrix} -1-\delta a - m_1 y_s & -m_1 x_s & 0 \\ m_1 y_s & m_1 x_s - 1 - \delta a & -m_2 y_s \\ 0 & 0 & m_2 y_s - 1 - \delta a \end{pmatrix} \Phi(t)$$

and $\Phi(0) = I$, the identity matrix. Hence the fundamental solution matrix is

$$\Phi(\tau) = \begin{pmatrix} \phi_{11}(\tau) \; \phi_{12}(\tau) & * \\ \phi_{21}(\tau) \; \phi_{22}(\tau) & ** \\ 0 \qquad 0 & \exp(\int_0^\tau (m_2 y_s(l) - 1)\mathrm{d}l) \end{pmatrix}. \tag{3.2}$$

It is no need to give the exact form of $(*)$ and $(**)$ as it is not required in the analysis that follows. The eigenvalues of the matrix $\Phi(\tau)$ are $\mu_3 = \exp(\int_0^\tau (m_2 y_s(l) - 1)\mathrm{d}l)$ and the eigenvalues μ_1, μ_2 of the following matrix

$$\begin{pmatrix} \phi_{11}(\tau) & \phi_{12}(\tau) \\ \phi_{21}(\tau) & \phi_{22}(\tau) \end{pmatrix}. \tag{3.3}$$

The μ_1, μ_2 are also the multipliers the locally linearizing system of the system (2.1) provided with $\frac{1}{\tau}\int_0^\tau m_1 \tilde{s}(l)\mathrm{d}l > 1$ at the asymptotically stable periodic solution $(x_s(t), y_s(t))$, according to theorem 2.1, we have that $\mu_1 < 1, \mu_2 < 1$.

If $m_1 > 1$ and $\frac{1}{\tau}\int_0^\tau m_2 y_s(l)\mathrm{d}l < 1$, the $\mu_3 = \exp(\int_0^\tau (m_2 y_s(l) - 1)\mathrm{d}l) < 1$, the boundary periodic solution $(x_s(t), y_s(t), 0)$ of the system (1.2) is locally asymptotically stable. We have that $z(t) \leqslant z(0)\exp(\int_0^t (m_2 y_s(l) - 1)\mathrm{d}l)$, hence we obtain that for any solution $(x(t), y(t), z(t))$ with $X(0) > 0$, $z(t) \to 0$ as $t \to \infty$. By $\lim_{t\to\infty} |x(t)+y(t)+z(t)-\tilde{s}(t)| = 0$, we have $\lim_{t\to\infty} |x(t)+y(t)-\tilde{s}(t)| = 0$. Now using theorem 2.1, we have $\lim_{t\to\infty} |y(t) - y_s(t)| = 0$ and $\lim_{t\to\infty} |x(t) - x_s(t)| = 0$.

If $m_1 > 1$ and $\frac{1}{\tau}\int_0^\tau m_2 y_s(l)\mathrm{d}l > 1$, the $\mu_3 = \exp(\int_0^\tau (m_2 y_s(l) - 1)\mathrm{d}l) > 1$, the boundary periodic solution $(x_s(t), y_s(t), 0)$ of the system (1.2) is unstable. We complete the proof. □

Let B denote the Banach space of continuous, τ-periodic functions N:$[0, \tau] \to R^2$. In the set B introduce the norm $|N|_0 = \sup_{0\leqslant t\leqslant\tau} |N(t)|$ with which B becomes a Banach space with the uniform convergence topology.

For convenience, we introduce the following lemmas 3.2 and 3.3 [11].

Lemma 3.2. [11]. Suppose $a_{ij} \in B$. (a) If $\int_0^\tau a_{22}(s)\mathrm{d}s \neq 0$, $\int_0^\tau a_{11}(s)\mathrm{d}s \neq 0$, then the linear homogenous system

$$\begin{aligned} \frac{\mathrm{d}y_1}{\mathrm{d}t} &= a_{11}y_1 + a_{12}y_2, \\ \frac{\mathrm{d}y_2}{\mathrm{d}t} &= a_{22}y_2 \end{aligned} \tag{3.4}$$

has no nontrivial solution in $B \times B$. In this case, the nonhomogeneous system

$$\begin{aligned} \frac{\mathrm{d}x_1}{\mathrm{d}t} &= a_{11}x_1 + a_{12}x_2 + f_1, \\ \frac{\mathrm{d}x_2}{\mathrm{d}t} &= a_{22}x_2 + f_2 \end{aligned} \tag{3.5}$$

has, for every $(f_1, f_2) \in B \times B$, a unique solution $(x_1, x_2) \in B \times B$ and the operator L: $B \times B \to B \times B$ defined by $(x_1, x_2) = L(f_1, f_2)$ is linear and compact. If we define that $x_2' = a_{22}x_2 + f_2$ has a unique solution $x_2 \in B$ and the operator $L_2\colon B \to B$ defined by $x_2 = L_2 f_2$ is linear and compact. Furthermore,

$x_1' = a_{11}x_1 + f_3$ for $f_3 \in B$ has a unique solution (since $\int_0^\tau a_{11}(s)\mathrm{d}s \neq 0$) in B and $x_1 = L_1 f_3$ defines a linear, compact operator L_1: $B \to B$. Then we have

$$L(f_1, f_2) \equiv (L_1(a_{12}L_2 f_2 + f_1), L_2 f_2). \tag{3.6}$$

(b) If $\int_0^\tau a_{22}(s)\mathrm{d}s = 0$, $\int_0^\tau a_{11}(s)\mathrm{d}s \neq 0$, then (3.4) has exactly one independent solution in $B \times B$.

Lemma 3.3. [11]. Suppose $a \in B$ and $\frac{1}{\tau}\int_0^\tau a(l)\mathrm{d}l = 0$. Then $x' = ax + f$, $f \in B$, has a solution $x \in B$ if and only if $\frac{1}{\tau}\int_0^\tau a(l)(\exp(-\int_0^l a(s)\mathrm{d}s))\mathrm{d}l = 0$.

By lemma 3.1, in its invariant manifold $\tilde{s} = x(t) + y(t) + z(t)$, the system (1.2) reduce to a equivalently nonautonomous system as following

$$\begin{aligned} &\frac{\mathrm{d}y}{\mathrm{d}t} = m_1(1 - y - z)y - (1 + \varepsilon a(t))y - m_2 yz,\\ &\frac{\mathrm{d}z}{\mathrm{d}t} = m_2 yz - (1 + \varepsilon a(t))z,\\ &y(0) > 0, z(0) \geqslant 0, y(0) + z(0) \leqslant \tilde{s}(0). \end{aligned} \tag{3.7}$$

If $m_1 > 1$, for the system (3.7), by theorem 3.1 the boundary periodic solution $(y_s(t), 0)$ is locally asymptotically stable provided with $m_2 < m_2^* := \frac{m_1}{m_1 - 1}$, and it is unstable provided with $m_2 > m_2^*$, hence the value m_2^* practises as a bifurcation threshold. For the system (3.7), we have the following results.

Theorem 3.2. For the system (3.7), $m_1 > 1$ is hold, then there exists a constance $\lambda_0 > 0$, such that for each $m_2 \in (m_2^*, m_2^* + \lambda_0)$, there exists a solution $(y, z) \in B \times B$ of (3.7) satisfying $0 < y < y_s, z > 0$, and $x = 1 - y - z > 0$ for all $t > 0$. Hence, the system (1.2) has a positive τ-periodic solution $(1 - y - z, y, z)$.

Proof. Let $x_1 = y - y_s(t), x_2 = z$ in (3.7), then

$$\begin{aligned} &\frac{\mathrm{d}x_1}{\mathrm{d}t} = -(2m_1 y_s + \varepsilon a)x_1 - (m_1 + m_2)y_s x_2 + g_1(x_1, x_2),\\ &\frac{\mathrm{d}x_2}{\mathrm{d}t} = (m_2 y_s - 1 - \varepsilon a)x_2 + g_2(x_1, x_2), \end{aligned} \tag{3.8}$$

where

$$g_1(x_1, x_2) = -m_1 x_1^2 - (m_1 + m_2)x_1 x_2, \quad g_2(x_1, x_2) = m_2 x_1 x_2.$$

We know that $\frac{1}{\tau}\int_0^\tau m_2 y_s(l)\mathrm{d}l - 1 \neq 0$, by lemma 3.3, using L we can equivalently write the system (3.8) as the operator equation

$$(x_1, x_2) = L^*(x_1, x_2) + G(x_1, x_2), \tag{3.9}$$

where

$$G(x_1, x_2) = (L_1(-(m_1 + m_2)y_s g_2(x_1, x_2) + g_1(x_1, x_2)), L_2 g_2(x_1, x_2)).$$

Here $L^*: B \times B \to B \times B$ is linear and compact and $G: B \times B \to B \times B$ is continuous and compact (since L_1 and L_2 are compact) and satisfies $G = o(|(x_1, x_2)|_0)$ near (0,0). A nontrivial solution $(x_1, x_2) \neq (0, 0)$ for some $m_2 > 1$ yields a solution $(y, z) = (y_s + x_1, x_2)$ of the system (3.7). Solutions $(y, z) \neq (y_s, 0)$ will be called nontrivial solutions of system (3.7).

We apply well-known local bifurcation techniques to (3.9). As is well known, bifurcation can occur only at the nontrivial solution of the linearized problem

$$(y_1, y_2) = L^*(y_1, y_2), m_2 > 0. \tag{3.10}$$

If $(y_1, y_2) \in B \times B$ is a solution of (3.10) for some $m_2 > 0$, then by the very manner in which L^* was defined, (y_1, y_2) solves the system

$$\begin{aligned} \frac{dy_1}{dt} &= -(2m_1 y_s + \varepsilon a)y_1 - (m_1 + m_2)y_s y_2, \\ \frac{dy_2}{dt} &= ((m_2 y_s - 1 - \varepsilon a)y_2 \end{aligned} \tag{3.11}$$

and conversely. Using lemma 3.3 (b), we see that (3.11) and hence (3.10) has one nontrivial solution in $B \times B$ if and only if $\frac{1}{\tau}\int_0^\tau m_2^* y_s(l)\,dl = 1$. Hence there exists a continuum $C = \{(m_2; x_1, x_2)\} \subseteq (0, \infty) \times B \times B$ nontrivial solutions of (3.10) such that the closure $\bar{C}$ contains $(m_2^*; 0, 0)$. This continuum gives rise to a continuum $C_1 = \{(m_2; y, z)\} \subseteq (0, \infty) \times B \times B$ of the solutions of (3.7) whose closure $\bar{C}_1$ contains the bifurcation point $(m_2^*; y_s, 0)$.

To see that solutions in C_1 correspond to solutions (y, z) of (3.7), we investigate the nature of the continuum C near the bifurcation point $(m_2^*; 0, 0)$ by expending m_2 and (x_1, x_2) in Lyapunov–Schmidt series:

$$\begin{aligned} m_2 &= m_2^* + \lambda\delta + \cdots, \\ x_1 &= x_{11}\delta + x_{12}\delta^2 + \cdots, \\ x_2 &= x_{21}\delta + x_{22}\delta^2 + \cdots \end{aligned}$$

for $x_{ij} \in B$ where δ is a small parameter. If we substitute these series into the differential system (3.7) and equate coefficients of δ and δ^2 we find that

$$\begin{aligned} x_{11}' &= -(2m_1 y_s + \varepsilon a)x_{11} - (m_1 + m_2^*)y_s x_{21}, \\ x_{21}' &= (m_2^* y_s - 1 - \varepsilon a)x_{21} \end{aligned}$$

and

$$x'_{12} = -(2m_1 y_s + \varepsilon a)x_{12} - (m_1 + m_2)y_s x_{22} + G_1(\lambda, x_{11}, x_{21}),$$
$$x'_{22} = (m_2^* y_s - 1 - \varepsilon a)x_{22} + x_{21}(\lambda y_s + m_2^* x_{11}),$$

respectively. Thus, $(x_{11}, x_{21}) \in B \times B$ must be a solution of (3.10). We choose the specific solution satisfying the initial conditions $x_{21}(0) = 1$. Then

$$x_{21} = \exp(\int_0^t (m_2^* y_s(l) - 1 - \varepsilon a(l))\mathrm{d}l) > 0.$$

Moreover $x_{11} < 0$ for all t. (This because $\int_0^\tau -2m_1 y_s \mathrm{d}l < 0$ implies that the Green's function for first equation in (3.11) is positive.) Using lemma 3.3 we find that

$$\lambda = -\frac{\int_0^\tau m_2^* x_{21}(l)x_{11}(l)\exp(-\int_0^l (m_2^* y_s(s) - 1 - \varepsilon a(s))\mathrm{d}s)\mathrm{d}l}{\int_0^\tau x_{21}(l)\exp(-\int_0^l (m_2^* y_s(s) - 1 - \varepsilon a(s))\mathrm{d}s)\mathrm{d}l} > 0.$$

Thus, we see that near the bifurcation point $(m_2^*; 0, 0)$ (say, for $0 < |m_2 - m_2^*| = \lambda|\varepsilon| < \lambda_0$) the continuum C has two (subcontinua) branches corresponding to $\varepsilon < 0, \varepsilon > 0$, respectively:

$$C^+ = \{(m_2; x_1, x_2) : m_2^* < m_2 < m_2^* + \lambda_0, x_1 < 0, x_2 > 0\},$$
$$C^- = \{(m_2; x_1, x_2) : m_2^* - \lambda_0 < m_2 < m_2^*, x_1 > 0, x_2 < 0\}.$$

The solution is on C^+ which prove the theorem, since $\lambda > 0$ is equivalent to $m_2 > m_2^*$. We have left only to show that $y = x_1 + y_s > 0$ for all t. This is easy, for if λ_0 is small, then y is near y_s in the sup norm of B; thus since y_s is bounded away from zero, so is y. At same time, by theorem 3.1, for the system (1.2), y is near y_s means that x is near x_s; thus $x = 1 - y - z > 0$. We notice that the periodic solution (y, z) is continuous $\tau-$periodic. So $x = 1 - y - z$ is continuous and $\tau-$periodic. We complete the proof.

4. Chemostat chaos

In this section, we will analyze the complexity of the periodic system (1.2). By theorems 2.1, 3.1, and 3.2, we know that if $m_1 < 1$, the periodic solution $(\tilde{s}(t), 0, 0)$ is globally asymptotically stable; if $m_1 > 1$ and $m_2 < m_2^* := \frac{m_1}{m_1 - 1}$, then the $(x_s(t), y_s(t), 0)$ is globally asymptotically stable. According to theorem 3.2, if $m_1 > 1$ and $m_2 > m_2^* := \frac{m_1}{m_1 - 1}$, the predator begins to invade the system. In the following we apply the forced model equations are

$$\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= (1 + \varepsilon \sin(t))(1 - x) - m_1 xy, \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= m_1 xy - (1 + \varepsilon \sin(t))y - m_2 yz, \\
\frac{\mathrm{d}z}{\mathrm{d}t} &= m_2 yz - (1 + \varepsilon \sin(t))z.
\end{aligned} \tag{4.1}$$

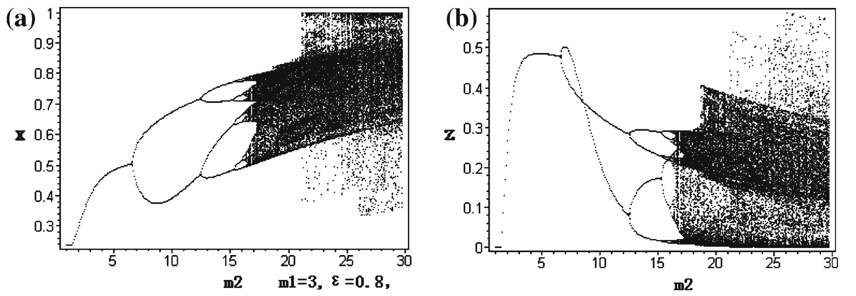

Figure 1. (a) (b) Bifurcation diagrams of *Poincaré* section for the substrate x and predator z in system (4.1) under $m_2 = 36$, $\varepsilon = 0.8$, and m_1 is varied.

We shall numerically integrate equation (4.1) and seek the long-term behavior of the solutions (after the transients have disappeared).

A traditional approach to gain preliminary insight into the properties of dynamic system is to carry out a one-dimensional bifurcation analysis. One-dimensional bifurcation diagrams of *Poincaré* maps provide information about the dependence of the dynamics on a certain parameter. The analysis is expected to reveal the type of attractor to which the dynamics will ultimately settle down after passing the initial transient phase and within which the trajectory will then remain forever.

First, we investigate the influence of m_2. In system (4.1), set $m_1 = 3, \varepsilon = 0.8, \tau = 2\pi$, and we choose $m_2 \in [0.2, 30]$ as the bifurcation parameter. The resulting bifurcation diagrams (figure 1) show: the invasion of predator at $m_2^* = 1.5$; by using theorem 3.1, when $m_2 > m_2^*$ to be not very large, the system shows stable period-one cycles [figure 2(a)]; as the parameter m_2 increases from 6.55, the period-one behavior bifurcates to a period-two cycles [figure 2(b)], after which period-doubling bifurcations (figure 2) ensure these culminate in a Feigenbaum cascade of period-doubling bifurcations leading to a chaotic region. A typical chaotic oscillation is captured when $m_2 = 18.8$ (figure 3). The main routs to chaos are Feigenbaum cascades. This periodic-doubling route to chaos is the hallmark of the logistic and Ricker maps [12, 13] and has been studied extensively by Mathematicians [14, 15].

Second, we want to investigate the influence of m_1. In system (4.1), set $m_2 = 20, \varepsilon = 0.8, \tau = 2\pi$, and we choose $m_1 \in [0.2, 6]$ as the bifurcation parameter. Figure 2 illustrates the bifurcation diagram of *Poincaré* map for equation (4.1). When m_1 is small ($m_1 < 1$), the solution $(1, 0, 0)$ is stable. When $m_1 > 1$, the prey begins invade the system and the solution $(x_s, y_s, 0)$ is stable if $m_1 < \frac{20}{19}$. When $m_1 > \frac{20}{19}$, the predator begins invade and a stable positive period solution is bifurcated from $(x_s, y_s, 0)$ if $m_1 < q_0 \approx 1.23$. However, when $m_1 > q_0$, the stability of $\tau-$periodic solution is destroyed and $2\tau-$periodic solution occurs and is stable if $m_1 < q_1 \approx 1.3$. When $m_1 > q_2 \approx 1.33$, it is unstable and there is a

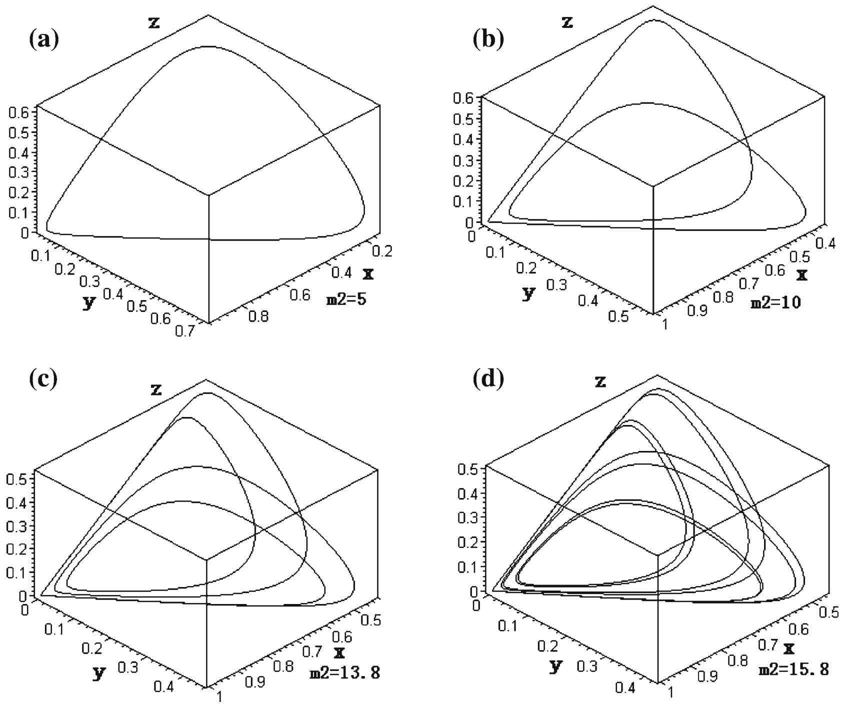

Figure 2. Periodic-doubling bifurcations. In equation (4.1), $m_1 = 3, \varepsilon = 0.8$ (a)–(d), are the complete trajectories of $\tau, 2\tau, 4\tau$, and 8τ-periodic solutions over the time interval from $t = 300\pi$ to $t = 500\pi$, corresponding with $m_2 = 5, 10, 13.8$ and 15.8.

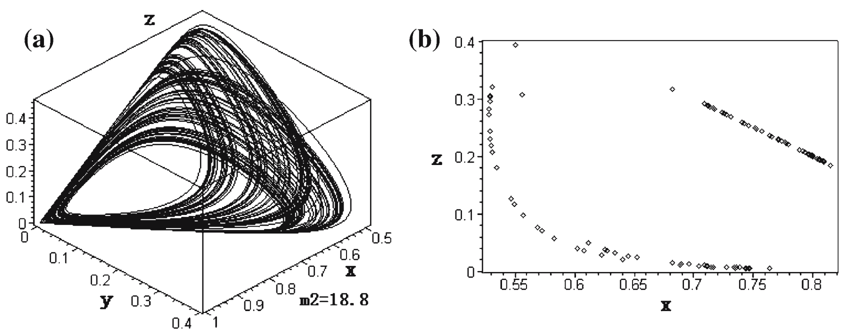

Figure 3. Strange attractors (chaos) of the flow by equation (4.1). Compare a Poincaré section (b) with the complete chaotic trajectory (a) ($m_2 = 18.8$). Poincaré points 150–250 are plotted in (b), and the corresponding complete trajectory over the time interval from $t = 300\pi$ to 500π are plotted in (a).

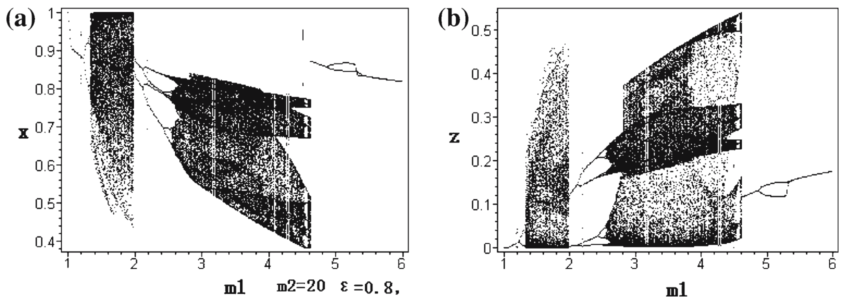

Figure 4. (a) (b) Bifurcation diagrams of *Poincaré* section for the substrate x and predator z in system (4.1) under $m_2 = 20$, $\varepsilon = 0.8$, and $m_1 \in [0.2, 6]$ is varied.

cascade of period doubling bifurcations leading to chaos. Continuously increasing $m_1 \approx 2$, the chaotic solution suddenly shrinks to a 2τ-period solution and further the system shows next doubling bifurcations and follows with a Feigenbaum cascade of period-doubling bifurcations leading to a chaotic region. When $m_2 > 4.63$, chaos suddenly shrinks $\tau-$periodic solutions.

Pitchfork bifurcations and tangent (saddle node) bifurcations are abundantly evident in cycles in figures 1 and 4, as well as attractor crises (the phenomenon of "crisis" in which chaotic attractors suddenly appear or disappear, or change size discontinuously as or change size discontinuously as a parameter smoothly varies, was first extensively analyzed by Grebogi et al. [16]). For instance, in figure 4, when m_2 is slightly increased beyond $m_2 = 2$ or 4.63, the chaotic attractor abruptly disappears, thus constituting a type of crisis.

5. Conclusions

In this paper, we introduce and study a model of Lotka-volterra chemostat food chain chemostat with periodically varying dilution rate, which contains with predator, prey, and substrate. First, we find the invasion threshold of the prey, which is $m_1^* = 1$. If $m_1 < m_1^*$, the periodic periodic solution $(\tilde{s}(t), 0, 0)$ is globally asymptotically stable and if $m_1 > m_1^*$, the prey starts to invade the system. Furthermore, by using Floquet theorem and small amplitude perturbation skills, we have proved that if $m_1 > m_1^*$, there exists $m_2^* = \frac{m_1}{m_1 - 1}$ to play as the invasion threshold of the predator, that is to say, if $m_2 < m_2^*$ the boundary solution $(x_s, y_s, 0)$ is globally asymptotically stable and if $m_2 > m_2^*$ the solution $(x_s, y_s, 0)$ is unstable. By using standard techniques of bifurcation theory, we prove that above this threshold there are periodic oscillations in substrate, prey, and predator.

Choosing different coefficients m_1 and m_2 as bifurcation parameters, we have obtained bifurcation diagrams (figures 1, 4). Bifurcation diagrams have shown that there exists complexity for system (1.2) including periodic doubling cascade. All these results show that dynamical behavior of system (1.2) becomes more complex with periodically varying dilution rate.

Acknowledgments

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